Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 20 — Sunday, October 21, 2018

1. **Problem:** Fluffy the Dog is an extremely fluffy dog. Because of his extreme fluffiness, children always love petting Fluffy anywhere. Given that Fluffy likes being petted 1/4 of the time, out of 120 random people who each pet Fluffy once, what is the expected number of times Fluffy will enjoy being petted?
   Solution: The expected number of times Fluffy will be petted is simply $\frac{1}{4} \times 120 = \boxed{30}$.

   Proposed by Annie Wang

2. **Problem:** Andy thinks of four numbers 27, 81, 36, and 41 and whispers the numbers to his classmate Cynthia. For each number she hears, Cynthia writes down every factor of that number on the whiteboard. What is the sum of all the different numbers that are on the whiteboard? (Don't include the same number in your sum more than once)

   Solution: Notice immediately that 27 is a multiple of 81 so they will share all divisors except 81. Next, notice that 36 shares every odd divisor with both 27 and 81, so we only need to add the even divisors of 36 to our total. Finally, notice that 41 only shares the divisor 1 with every other number. Our final total is $(1+3+9+27+81)+(2+4+6+12+18+36)+41 = \boxed{240}$

   Proposed by Eddie Wang

3. **Problem:** Charles wants to increase the area his square garden in his backyard. He increases the length of his garden by 2 and increases the width of his garden by 3. If the new area of his garden is 182, then what was the original area of his garden?

   Solution: Let the original side length of the square be s. We know that the length of the new rectangle is $s+2$, and the width of the rectangle is $s+3$. Therefore,

   $$(s+2)(s+3) = 182$$
   $$\Rightarrow\ s^2+5s+6 = 182$$
   $$\Rightarrow\ s^2+5s-176 = 0$$
   $$\Rightarrow\ (s+16)(s-11) = 0$$
   $$\Rightarrow\ s = 11, -16$$

   Clearly, the side length of a square cannot be negative, so $s = 11$. Thus, the original area of the garden is $11^2 = \boxed{121}$

   Alternative solution: Following the above solution to $(s+2)(s+3) = 182$, we finish using the well known guess and check method. We see that $182 = 13 \cdot 14$, so $s+2 = 13$ or $s = 11$. Then $11^2 = 121$.

   Proposed by Eddie Wang

4. **Problem:** Antonio is trying to arrange his flute ensemble into an array. However, when he arranges his players into rows of 6, there are 2 flute players left over. When he arranges his players into rows of 13, there are 10 flute players left over. What is the smallest possible number of flute players in his ensemble such that this number has three prime factors?

   Solution: A simple way to approach this problem is by examining every number that will leave a remainder of 10 when divided by 13 until we find one that also will leave a remainder of 2 when divided by 6. To make things simpler, we can ignore all odd numbers because no odd number will leave a remainder of 2 when divided by 6. We find that the smallest such number is 62; however, 62 only has 2 prime factors. To find the next such smallest number, we can simply add 78 which gets us 140. $140 = 2 \times 5 \times 7$ so it does indeed have three prime factors. Therefore, our answer is $\boxed{140}$ flute players.

   Proposed by Poonam

5. **Problem:** On the AMC 9 (Acton Math Competition 9), 5 points are given for a correct answer, 2 points are given for a blank answer and 0 points are given for an incorrect answer. How many possible scores are there on the AMC 9, a 15 problem contest?

   Solution: The lowest possible score is 0, and the highest is 75. We can write all numbers as $5n, 5n+1, 5n+2, 5n+3$, or $5n+4$ for some integer n. We can clearly get all scores of the form $5n$. For $5n+2$, we achieve the score for $0 \le n \le 14$, because we need to leave one blank. For $5n+4$, we only get $0 \le n \le 13$. For $5n+1 = 5(n-1)+6$ we must leave 3 blank, so we get $0 \le n-1 \le 12$ or $1 \le n \le 13$. For $5n+3$ we get $0 \le n-1 \le 11$ or $1 \le n \le 12$. In total, we have $16+15+14+13+12 = \boxed{70}$ possible scores.

   Proposed by Aaron Zhang

6. **Problem:** Charlie Puth produced three albums this year in the form of CD's. One CD was circular, the second CD was in the shape of a square, and the final one was in the shape of a regular hexagon. When his producer circumscribed a circle around each shape, he noticed that each time, the circumscribed circle had a radius of 10. The total area occupied by 1 of each of the different types of CDs can be expressed in the form $a + b\pi + c\sqrt{d}$ where d is not divisible by the square of any prime. Find $a+b+c+d$.

   Solution: We will find the areas of each of the three CDs separately. The area of the first CD can be calculated immediately to be 100π, since we know its radius is 10. For the second CD, we know that the circumscribed circle's radius must be half the length of the square's diagonal. Therefore, the diagonal of the square must have length 20, and the length of one of its sides must have length $\frac{20}{\sqrt{2}} = 10\sqrt{2}$. Thus, the area of the second CD is $(10\sqrt{2})^2 = 200$. For the third CD, we can break up the regular hexagon into six congruent equilateral triangles. The side length of each of these triangles must be the radius of the circumscribed circle which is 10. Thus, the area of all six triangles is $6(\frac{10^2\sqrt{3}}{4}) = \frac{600\sqrt{3}}{4} = 150\sqrt{3}$. Putting together all three areas in the required form, we have that the sum of the areas is equal to $200 + 100\pi + 150\sqrt{3}$. Thus, the answer is $200 + 100 + 150 + 3 = \boxed{453}$

   Proposed by Eddie Wang

7. **Problem:** You are picking blueberries and strawberries to bring home. Each bushel of blueberries earns you 10 dollars and each bushel of strawberries earns you 8 dollars. However your cart can only fit 24 bushels total and has a weight limit of 100 lbs. If a bushel of blueberries weighs 8 lbs and each bushel of strawberries weighs 6 lbs, what is your maximum profit. (You can only pick an integer number of bushels)

   Solution: Let b, s be the bushels of blueberries and strawberries you pick, respectively. We have $b + s \le 24$ and $8b + 6s \le 100$. We want to maximize $10b + 8s$. We can graph this on the Cartesian plane. The boundary points are $(b, s) = (11, 2)$ and $(2, 14)$. The first gives $110 + 16 = \$126$ while the second gives $20 + 112 = \$132$, so the maximum profit is $\boxed{132}$.

   Proposed by Aaron Zhang

8. **Problem:** The number

   $$\sqrt{2218 + 144\sqrt{35} + 176\sqrt{55} + 198\sqrt{77}}$$

   can be expressed in the form $a\sqrt{5} + b\sqrt{7} + c\sqrt{11}$ for positive integers a, b, c. Find abc.

   Solution: Squaring both expression, we get $2218 + 144\sqrt{35} + 176\sqrt{55} + 198\sqrt{77} = 5a^2 + 7b^2 + 11c^2 + 2ab\sqrt{35} + 2ac\sqrt{55} + 2bc\sqrt{77}$. Matching coefficients, we get $2ab = 144, 2ac = 176, 2bc = 198$ so $ab = 72, ac = 88, bc = 99$ from which we easily guess $a = 8, b = 9, c = 11$. Then $abc = \boxed{792}$.

   Proposed by Poonam

9. **Problem:** Let (x, y) be a point such that no circle passes through the three points $(9, 15), (12, 20), (x, y)$, and no circle passes through the points $(0, 17), (16, 19), (x, y)$. Given that $x - y = -\frac{p}{q}$ for relatively prime positive integers p, q, Find $p + q$.

   Solution: The only possible way for this to occur is if both sets of three points form a line, since there always exists a circle that passes through three noncollinear points in a plane. Using point-slope form, the equation of the first line that passes through $(9, 15)$ and $(12, 20)$ is $y - 15 = \frac{20-15}{12-9}(x - 9)$ or $y - 15 = \frac{5}{3}(x - 9)$ while the equation second line that passes through $(0, 17)$ and $(16, 19)$ is $y - 17 = \frac{19-17}{16-0}(x - 0)$ or $y - 17 = \frac{1}{8}x$. We can simply solve a system of these 2 equations to find x and y:

$$\begin{cases} y - 15 = \frac{5}{3}(x - 9) \\ y - 17 = \frac{1}{8}x \end{cases}$$

   Solving, we find that $x = \frac{408}{37}$ and $y = \frac{680}{37}$. Then $\frac{408}{37} - \frac{680}{37} = -\frac{272}{37}$, so $p + q = 272 + 37 = \boxed{309}$.

   Proposed by Advay

10. **Problem:** How many ways can Alfred, Betty, Catherine, David, Emily and Fred sit around a 6 person table if no more than three consecutive people can be in alphabetical order (clockwise)?

   Solution: First, note that I will be referring to these people as A, B, C, D, E, and F for the remainder of this solution.

   A circle is kind of annoying, so let's just focus in on an easier perspective. Let us consider this seating arrangement as a line that starts with A.

   We use complementary counting, taking away the cases with 4, 5, or 6 in row in alphabetical order.

   For 6 in a row: obviously it has to be ABCDEF so 1 way.

   For 5 in a row: It must be the first 5 in alph-order and the sixth one not. If the last letter is B,C,D, or E there is one way to arrange the other five in order(ACDEFB, ABDEFC, ABCEFD, or ABCDFE). Thus 4 ways.

   For 4 in a row: either first four or last four must be in alph-order because the letter after A is always alphabetically after A. As long as B is not he second letter, we can have the last four be in alph-order without having all 6 being in order, so there ware 4 ways for this subcase. Then, for the subcase with the first four in alph-order, we have 5 C 2 = 10 ways to choose the last two letters. EF doesn't work, and for DF, CF, BF, there is only one way to arrange them (eg ABCEDF while ABCEFD doesn't work) but for the other 6 we can swap the last two letters (ABCFDE and ABCFED both work), so we have 3+6*2 = 15 ways. Then in total we have 1+4+4+15 = 24 cases with 4,5,or 6 in a row, so then the answer is 120 - 24 = $\boxed{96}$.

   Proposed by Justin