Acton-Boxborough Math Competition Online Contest Solutions

Saturday, November 16 — Sunday, November 17, 2019

1. **Problem:** The remainder of a number when divided by 7 is 5. If I multiply the number by 32 and add 18 to the product, what is the new remainder when divided by 7?

   Solution: We can express the number as $7x + 5$ where x is an integer. Then, multiplying by 32 and adding 18 results in $(7x + 5)32 + 18 = 224x + 178$. We already know that $7|224x$, and we have that $178 \equiv 3 \pmod{7}$. Then the new remainder is $\boxed{3}$.

   Proposed by Eddie Wang

2. **Problem:** A fair coin is flipped 15 times. If the probability that there are more heads than tails is $\frac{a}{b}$, find $a + b$.

   Solution: Notice that the cases where more heads are flipped than tails are exactly symmetrical to the cases where more tails are flipped than heads (i.e. 15 heads corresponds to 15 tails, 14 heads and 1 tail corresponds to 14 tails and 1 head...8 heads and 7 tails corresponds to 8 tails and 7 heads). Each corresponding case has the same probability of occurring (i.e. the probability of x heads and y tails has the same chance of occurring as y heads and x tails). Thus, the probability is just $\frac{1}{2}$ so our answer is $\boxed{3}$

   Proposed by Aaron Zhang

3. **Problem:** Let $-\frac{\sqrt{p}}{q}$ be the smallest nonzero real number such that the reciprocal of the number is equal to the number minus the square root of the square of the number, where p and q are positive integers and p is not divisible the square of any prime. Find $p + q$.

   Solution: If we let the number be x, the equation is $\frac{1}{x} = x - \sqrt{x^2}$. Then $\sqrt{x^2}$ is the same as $|x|$, so we have $\frac{1}{x} = x - |x|$. If x is positive, then the right side is 0. If x is negative, $\frac{1}{x} = 2x \Rightarrow\ 1 = 2x^2 \Rightarrow\ x = \pm\frac{\sqrt{2}}{2}$ so $x = -\frac{\sqrt{2}}{2}$ so the answer is $\boxed{4}$.

   Proposed by Jerry Tan

4. **Problem:** Rachel likes to put fertilizers on her grass to help her grass grow. However, she has cows there as well, and they eat 3 little fertilizer balls on average. If each ball is spherical with a radius of 4, then the total volume that each cow consumes can be expressed in the form $a\pi$ where a is an integer. What is a?

   Solution: The formula for the volume of a sphere is $\frac{4\pi r^3}{3}$, so the volume of each fertilizer ball with radius 4 is $\frac{4^4\pi}{3} = \frac{256\pi}{3}$. Since each cow consumes 3 fertilizer balls, the total volume that each cow consumes is 256π. Thus a is $\boxed{256}$.

   Proposed by Eddie Wang

5. **Problem:** One day, all 30 students in Precalc class are bored, so they decide to play a game. Everyone enters into their calculators the expression $9 \diamondsuit 9 \diamondsuit 9 ... \diamondsuit 9$, where 9 appears 2020 times, and each $\diamondsuit$ is either a multiplication or division sign. Each student chooses the signs randomly, but they each choose one more multiplication sign than division sign. Then all 30 students calculate their expression and take the class average. Find the expected value of the class average.

   Solution: Given in the problem that there is always 1 more multiplication sign than division sign, the value of the expression will always be 81, regardless of the combination of multiplication or division signs. The expected value is thus also $\boxed{81}$.

   Proposed by Jerry Tan

6. **Problem:** NaNoWriMo, or National Novel Writing Month, is an event in November during which aspiring writers attempt to produce novel-length work–formally defined as 50,000 words or more–within the span of 30 days. Justin wants to participate in NaNoWriMo, but he's a busy high school

student: after accounting for school, meals, showering, and other necessities, Justin only has six hours to do his homework and perhaps participate in NaNoWriMo on weekdays. On weekends, he has twelve hours on Saturday and only nine hours on Sunday, because he goes to church. Suppose Justin spends two hours on homework every single day, including the weekends. On Wednesdays, he has science team, which takes up another hour and a half of his time. On Fridays, he spends three hours in orchestra rehearsal. Assume that he spends all other time on writing. Then, if November 1st is a Friday, let w be the minimum number of words per minute that Justin must type to finish the novel. Round w to the nearest whole number.

Solution: November has 30 days, or 4 full weeks and two days ($30 - 4 * 7 = 2$). For each complete week, Justin has $6 * 5 + 9 + 12 = 51$ hours. Then, subtracting homework, science team, and orchestra, Justin has $51 - 2 * 7 - 1.5 - 3 = 32.5$ hours left. For four weeks, that is $32.5 * 4 = 130$ hours. Since the month starts on a Friday and 30 mod 7 = 2, November must end on a Sunday, making the extra two days a Saturday and a Sunday. Subtract 2 hours of homework from each of those days, there are $12 + 9 - 2 * 2 = 17$ hours. Thus, in total, Justin has $130 + 17 = 147$ hours or $147 * 60 = 8820$ minutes. The speed then must be $50,000/8820 \approx 5.67 \approx \boxed{6}$ words per minute

Proposed by Poonam Sahoo

7. **Problem:** Let positive reals a, b, c be the side lengths of a triangle with area 2030. Given $ab + bc + ca = 15000$ and $abc = 350000$, find the sum of the lengths of the altitudes of the triangle.

Solution: Let h_a, h_b, h_c be the altitudes to sides a, b, c respectively. Then for each side length we have that the height multiplied by the base is $2030 \cdot 2 = 4060$. Then $h_a = 4060/a, h_b = 4060/b, h_c = 4060/c$, so the sum of the lengths of the altitudes is $4060(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$. Then using the given equations $ab + bc + ca = 15000$ and $abc = 350000$ we have that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{15000}{350000} = \frac{3}{70}$. Then

$$h_a + h_b + h_c = 4060\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 4060 \cdot \frac{3}{70} = \boxed{174}.$$

Proposed by Jerry Tan

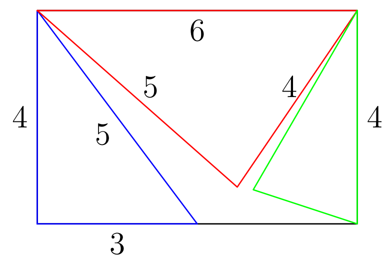
8. **Problem:** Find the minimum possible area of a rectangle with integer sides such that a triangle with side lengths 3,4,5, a triangle with side lengths 4,5,6, and a triangle with side lengths $\frac{9}{4}$,4,4 all fit inside the rectangle without overlapping.

Solution: Let the area of the rectangle be K. First we find the sum of areas of the triangles using Heron's. A 3-4-5 triangle has area 6; A 4-5-6 triangle has area $\sqrt{\frac{15 \cdot 7 \cdot 5 \cdot 3}{16}} = \frac{15\sqrt{7}}{4}$. Drawing the altitude in a $\frac{9}{4}$-4-4 triangle and using Pythagorean Theorem gives area $\frac{9\sqrt{943}}{64}$. Since $30.5^2 = 930.25$, $\sqrt{943} > \frac{61}{2}$ so $\frac{9\sqrt{943}}{64} > \frac{549}{128} = 4\frac{37}{128} > 4.25$. We have $175 > 169$ so $5\sqrt{7} > 13$ so $15\sqrt{7} > 39$ which means $\frac{15\sqrt{7}}{4} > 9.75$. Then the sum of the 3 areas is $> 6 + 4.25 + 9.75 = 20$. Thus the rectangle must have area more than 20.

Now consider the 4-5-6 triangle. The shortest altitude is $> 2 \cdot 9.75/6 > 3$. If the length or width of the rectangle is less than or equal to 3, there is no way to fit the triangle because it will take up at least the length of its shortest altitude, no matter its orientation. Thus both the length and width of the rectangle must be at least 4.

Now, if $K = 21, 22$, or 23, one side is not at least 4. However for $K = 24$ satisfies this. We just need to check that we can actually fit the triangle in a 4 by 6 rectangle.

One possible configuration is shown below. Consider a 4 by 6 rectangle on the coordinate plane with vertices at $(0,0), (6,0), (6,4), (0,4)$. We can put the 3-4-5 triangle at vertices $(0,0), (3,0), (0,4)$. Then we place the 4-5-6 with the side of length 6 along the segment $(0,4)$ to $(6,4)$ with the side of length 5 having an endpoint at $(0,4)$. By Pythagorean we find the coordinate of the third vertex to be $(\frac{15}{4}, 4-\frac{5\sqrt{7}}{4})$. We can check that both angles of the two triangles at $(0,4)$ are less than 45 so they don't overlap. Finally, we must check that the distance from $(\frac{15}{4}, 4-\frac{5\sqrt{7}}{4})$ to $(6,0)$ is greater than $\frac{9}{4}$, which is obvious since the x-coordinates differ by $\frac{9}{4}$. Thus the answer is $\boxed{24}$.



Proposed by Jerry Tan

9. **Problem:** The base 16 number $10111213...99_{16}$, which is a concatenation of all of the (base 10) 2-digit numbers, is written on the board. Then, the last $2n$ digits are erased such that the base 10 value of remaining number is divisible by 51. Find the smallest possible integer value of n.

   Solution: Notice that $51 = 17 \times 3$. In base 16, a number that is divisible by 17 must have alternating digit sums differ by a multiple of 17 (similar to divisibility by 11 in base 10). In addition, a number that is divisible by 3 in base 16 must have a digit sum divisible by 3 (same as divisibility by 3 in base 10). We will use these two facts to help us.

   The difference of the alternating digit sums is $9 \times (0+1+2+\cdots+9) - 10 \times (1+2+\cdots+9) = -45$ which corresponds to $6 \pmod{17}$. Our goal is to make this $0 \pmod{17}$ after removing the smallest number of pairs of digits.

   We notice immediately that after removing the last 8 pairs of digits or last 16 digits, the difference of the alternating digit sums will be divisible by 17. Now it suffices to check whether the remaining number $10111213...91_{16}$ will be divisible by 3 by checking its digit sum. The digit sum is $10 \times (1+2+\cdots+8) + 8 \times (0+1+2+\cdots+9) + 9 + 0 + 9 + 1 = 10 \times 36 + 8 \times 45 + 19$ which is not divisible by 3, so we have to keep removing digits.

   We then notice that after removing another 2 pairs of digits or 4 more digits, the difference of the alternating digit sums will be divisible by 17 again. Now it suffices to check whether the remaining number $10111213...89_{16}$ will be divisible by 3 by checking its digit sum. The digit sum is $10 \times (1+2+\cdots+8) + 8 \times (0+1+2+\cdots+9) = 10 \times 36 + 8 \times 45$ which is divisible by 3. Thus, $10111213...89_{16}$ will be divisible by 51 when expressed in base 10. We erased 10 pairs of digits, so $n = \boxed{10}$.

   Proposed by Jerry Tan

10. **Problem:** Consider sequences that consist entirely of X's, Y's and Z's where runs of consecutive X's, Y's, and Z's are at most length 3. How many sequences with these properties of length 8 are there?

    Solution: Let the number of sequences of length n with the above properties be $F(n)$. When $n \leq 3$, there are no restrictions for which letters can be used, so $F(1) = 3^1 = 3$, $F(2) = 3^2 = 9$, and $F(3) = 3^3 = 27$. From here, we construct a recursion for greater n. The last run can not be made from the same letter as the previous run, so there are two choices for the letter of the last run. Also, the last run can have a length of 1, 2, or 3, so we have the recursion $F(n) = 2F(n-1) + 2F(n-2) + 2F(n-3)$. Plugging in $n = 8$ gives us that there are $F(8) = \boxed{5676}$ sequences of length 8 with the aforementioned properties.

    Proposed by Annie Wang